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Diffusion in flashing periodic potentials

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Abstract. The one-dimensional overdamped Brownian motion in a symmetric periodic potential modulated by external time-reversible noise is analyzed. The calculation of the effective diffusion coefficient is reduced to the mean first passage time problem. We derive general equations to calculate the effective diffusion coefficient of Brownian particles moving in arbitrary supersymmetric potential modulated by: (i) an external white Gaussian noise and (ii) a Markovian dichotomous noise. For both cases the exact expressions for the effective diffusion coefficient are derived. We obtain acceleration of diffusion in comparison with the free diffusion case for fast fluctuating potentials with arbitrary profile and for sawtooth potential in case (ii). In this case the parameter region where this effect can be observed is given. We obtain also a finite net diffusion in the absence of thermal noise. For rectangular potential the diffusion slows down, for all parameters of noise and of potential, in comparison with the case when particles diffuse freely.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.-r Probability theory, stochastic processes, and statistics – 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.)

1 Introduction

Recently a considerable amount of analysis has been devoted to investigate the transport of Brownian particles in spatially periodic stochastic structures, such as Josephson junctions [1], Brownian motors [2] and molecular motors [3]. Specifically there has been great interest in studying influences of symmetric forces on transport properties, and in calculating the effective diffusion coefficient in the overdamped limit in particular [2,4–7]. Analytical results were obtained in arbitrary fixed periodic potential, tilted periodic potentials, symmetric periodic potentials modulated by white Gaussian noise, and in supersymmetric potentials [4–8,10,11]. The acceleration of diffusion in comparison with the free diffusion was obtained in references [4,6,7,11]. At thermal equilibrium there is not net transport of Brownian particles, while away from equilibrium, the occurrence of a current (*ratchet effect*) is observed generically. Therefore, the absence rather the presence of net flow of particles in spite of a broken spatial symmetry is the very surprising situation away from thermal equilibrium, as stated in references [2,5]. Moreover, the problem of the sorting of Brownian particles by the enhancement of their effective diffusion coefficient has been increasingly investigated in the last years, both from experimental [12–14] and theoretical points of view [4,15]. Specifically, the enhancement of diffusion in *symmetric* potentials was investigated in references [4,14].

Motivated by these studies and by the problem of dopant diffusion acceleration in semiconductors physics [16,17], we try to understand how nonequilibrium symmetrical correlated forces influence thermal systems when potentials are symmetric, and if there are new features which characterize the relaxation process in symmetric potentials. This is done by using a fluctuating periodic potential satisfying the supersymmetry criterion [5], and a different approach with respect to previous theoretical investigations (see review of Reimann in Ref. [2]). Using the analogy between a continuous Brownian diffusion at large times and the "jump diffusion" model [10,11], we reduce the calculation of effective diffusion coefficient D_{eff} to the first passage time problem. We consider potentials modulated by external white Gaussian noise and by Markovian dichotomous noise. For the first case we derive the exact formula of D_{eff} for arbitrary potential profile. The general equations obtained for randomly switching potential are solved for the sawtooth and rectangular periodic potentials, and the exact expression of D_{eff} is derived without any assumptions on the intensity of driving white Gaussian noise and switchings mean rate of the potential.

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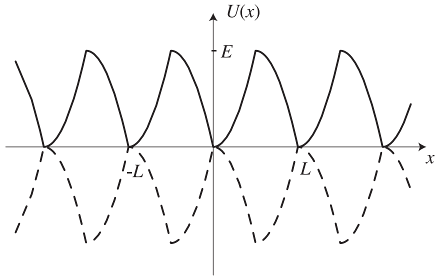

Fig. 1. Periodic potential with supersymmetry.

2 Fast fluctuating periodic potential

The effective diffusion coefficient in a fast fluctuating sawtooth potential was first investigated and derived in reference [6]. In papers [7] we generalized this result to the case of arbitrary potential profiles. We consider the following Langevin equation

$$\frac{dx}{dt} = -\frac{dU\left(x\right)}{dx} \cdot \eta\left(t\right) + \xi\left(t\right), \qquad (1)$$

where $x(t)$ is the displacement in time t, $\xi\left(t\right)$ and $\eta\left(t\right)$ are statistically independent Gaussian white noises with zero means and intensities $2D$ and $2D_\eta$, respectively. Further we assume that the potential $U\left(x\right)$ satisfies the supersymmetry criterion [5]

$$E - U\left(x\right) = U\left(x - \frac{L}{2}\right), \qquad (2)$$

where L is the spatial period of the potential (see Fig. 1). Following reference [8] and because we have $\langle x\left(t\right)\rangle = 0$, we determine the effective diffusion coefficient as the limit

$$D_{eff} = \lim_{t\to\infty} \frac{\left\langle x^2(t)\right\rangle}{2t}. \qquad (3)$$

To calculate the effective diffusion constant we use the "jump diffusion" model [10,11]

$$\tilde{x}(t) = \sum_{i=1}^{n(0,t)} q_i, \qquad (4)$$

where q_i are statistically independent random increments of jumps with values $\pm L$ and $n(0,t)$ denotes the total number of jumps in the time interval $(0,t)$. In the asymptotic limit $t\to\infty$, the "fine structure" of a diffusion is unimportant, and the random processes $x\left(t\right)$ and $\tilde{x}(t)$ become statistically equivalent, therefore $\left\langle x^2\left(t\right)\right\rangle \simeq \left\langle \tilde{x}^2(t)\right\rangle$. Because of the supersymmetry of the potential $U\left(x\right)$, the probability density reads $P\left(q\right) = \left[\delta\left(q-L\right)+\delta\left(q+L\right)\right]/2$. From equations (3) and (4) we arrive at

$$D_{eff} = \frac{L^2}{2\tau}, \qquad (5)$$

where $\tau = \langle\tau_j\rangle$ is the mean first-passage time (MFPT) for Brownian particle with initial position $x=0$ and absorbing boundaries at $x = \pm L$. In fluctuating periodic potentials, therefore, the calculation of D_{eff} reduces to the MFPT problem. Solving the equation for the MFPT of Markovian process $x(t)$ we obtain the exact formula for D_{eff}

$$D_{eff} = D\left[\frac{1}{L}\int_0^L \frac{dx}{\sqrt{1+D_\eta\left[U'\left(x\right)\right]^2/D}}\right]^{-2}. \qquad (6)$$

From equation (6), $D_{eff} > D$ for an arbitrary potential profile $U\left(x\right)$, therefore we have always the enhancement of diffusion in comparison with the case $U\left(x\right) = 0$. We emphasize that the value of diffusion constant does not depend on the height of potential barriers, as for fixed potential [8], but it depends on its gradient $U'\left(x\right)$. The dependencies of effective diffusion constant D_{eff} on the intensity D_η of the modulating white noise are plotted in Figure 2 for sawtooth, sinusoidal and piece-wise parabolic potential profiles.

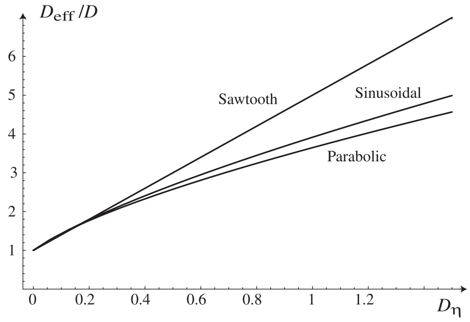

Fig. 2. Enhancement of diffusion in fast fluctuating periodic potentials.

3 Randomly switching periodic potential profile

Now we consider equation (1) where $\eta(t)$ is a Markovian dichotomous noise, which takes the values ±1 with switchings mean rate ν. Thus, we investigate the Brownian diffusion in a supersymmetric periodic potential flipping between two configurations, $U\left(x\right)$ and $-U\left(x\right)$. In the "overturned" configuration the maxima of the potential become the minima and vice versa. In accordance with equation (2), we can rewrite equation (1) as

$$\frac{dx}{dt} = -\frac{\partial}{\partial x} U\left(x + \frac{L}{4}\left[\eta\left(t\right)-1\right]\right) + \xi\left(t\right), \qquad (7)$$

Because of supersymmetric potential and time-reversible Markovian dichotomous noise the ratchet effect is absent: $\langle\dot{x}\rangle = 0$. All Brownian particles are at the origin at $t=0$ and the "jump diffusion" model (4) and (5) is used, because the non-Markovian process $x\left(t\right)$ has Markovian dynamics between flippings. The probability density of random increments q_i is the same of previous case and the distribution of waiting times t_j reads

$$w\left(t\right) = \frac{w_+\left(t\right)+w_-\left(t\right)}{2}, \qquad (8)$$

where $w_+(t)$ and $w_-(t)$ are the first passage time distributions for the configuration of the potential with $\eta(0) = +1$ and $\eta(0) = -1$ respectively. In accordance with equation (8), τ is the semi-sum of the MFPTs τ_+ and τ_-, corresponding to the probability distributions $w_+(\tau)$ and $w_-(\tau)$. The exact equations for the MFPTs of Brownian diffusion in randomly switching potentials, derived from the backward Fokker-Planck equation, are

$$\begin{aligned} &D\tau_+'' - U'(x)\,\tau_+' + \nu(\tau_- - \tau_+) = -1,\\ &D\tau_-'' + U'(x)\,\tau_-' + \nu(\tau_+ - \tau_-) = -1, \end{aligned} \qquad (9)$$

where $\tau_+(x)$ and $\tau_-(x)$ are the MFPTs for initial values $\eta(0) = +1$ and $\eta(0) = -1$, respectively, with the starting position of the Brownian particle at the point x. We consider the initial position at $x = 0$ and solve equations (9) with the absorbing boundaries conditions $\tau_\pm(\pm L) = 0$. Because of the symmetry of the potential and of the dichotomous noise, the probability flow at the point $x = 0$ equals zero at any times. We solve, therefore, the equations (9) in the range $(0, L)$ with the following boundary conditions [18]: $\tau_\pm'(0) = 0, \quad \tau_\pm(L) = 0$. By introducing two auxiliary functions

$$T(x) = \frac{\tau_+(x) + \tau_-(x)}{2}, \quad \theta(x) = \frac{\tau_+(x) - \tau_-(x)}{2} \qquad (10)$$

we can rewrite equations (9) as

$$\begin{aligned} &T'' - \frac{U'(x)}{D}\,\theta' = -\frac{1}{D},\\ &\theta'' - \frac{U'(x)}{D}\,T' - \frac{2\nu}{D}\,\theta = 0. \end{aligned} \qquad (11)$$

After integrating the first equation (11) in the interval $(0, x)$ and substituting the function obtained T'

$$T'(x) = -\frac{x}{D} + \frac{1}{D}\int_0^x U'(y)\,\theta'(y)\,dy, \qquad (12)$$

into the second equation (11), we obtain the following integro-differential equation for the function $\theta(x)$

$$\theta'' - \frac{U'(x)}{D^2}\int_0^x U'(y)\,\theta'(y)\,dy - \frac{2\nu}{D}\,\theta = -\frac{xU'(x)}{D^2}. \qquad (13)$$

After integrating equation (12) in the interval $(0, L)$, with the above-mentioned boundary conditions and using equation (5), we get finally

$$\frac{D_{\mathit{eff}}}{D} = \left[1 - \frac{2}{L}\int_0^L \left(1 - \frac{x}{L}\right) U'(x)\,\theta'(x)\,dx\right]^{-1}. \qquad (14)$$

The general equations (13) and (14) solve formally the problem to calculate the effective diffusion coefficient in the supersymmetric periodic potential $U(x)$.

3.1 Switching sawtooth periodic potential

In such a case (see Fig. 3) from equations (13) and (14), after algebraic rearrangements, we obtain the following exact result

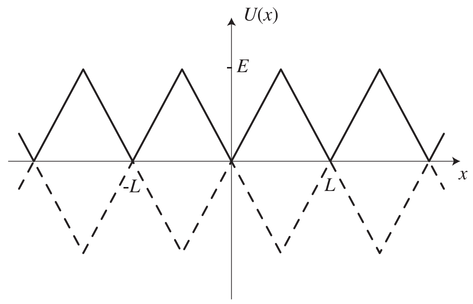

Fig. 3. Switching sawtooth periodic potential.

$$\begin{aligned} \frac{D_{\mathit{eff}}}{D} = {} & 2\alpha^2(1+\mu)\left(A_\mu + \mu + \mu^2\cosh 2\alpha\right)\\ & \times \Big(2\alpha^2\mu^2(1+\mu) + 2\mu A_{\mu 1}\sinh^2\alpha\\ & + 4\alpha\mu A_\mu \sinh\alpha + 8A_{\mu 2}\sinh^2(\alpha/2)\Big)^{-1}, \end{aligned} \qquad (15)$$

where $A_\mu = 1 - 3\mu + 4\mu\cosh\alpha$, $A_{\mu 1} = 7 - \mu + 2\alpha^2\mu^2$, and $A_{\mu 2} = 1 - 6\mu + \mu^2$. Here $\alpha = \sqrt{(E/D)^2 + \nu L^2/(2D)}$ and $\mu = \nu L^2 D/(2E^2)$ are dimensionless parameters, E is the potential barrier height. The equation (15) was derived without any assumptions on the intensity of white Gaussian noise, the mean rate of switchings and the values of the potential profile parameters. We introduce two new dimensionless parameters with a clear physical meaning: $\beta = E/D$, which is the ratio between the height of the potential barrier and the intensity of white Gaussian noise, and $\omega = \nu L^2/(2D)$, which is the ratio between the free diffusion time through the distance L and the mean time interval between switchings. The parameters α and μ can be expressed in terms of β and ω as: $\alpha = \sqrt{\beta^2 + \omega}$, $\mu = \omega/\beta^2$. Let us analyze the limiting cases. At very rare flippings ($\omega \to 0$) we have $\alpha \simeq \beta$, $\mu \to 0$ and equation (15) gives

$$\frac{D_{\mathit{eff}}}{D} \simeq \frac{\beta^2}{4\sinh^2(\beta/2)}, \qquad (16)$$

which coincides with the result obtained for the fixed periodic potential. For very fast switchings ($\omega \to \infty$) the Brownian particles "*see*" the average potential, i.e. $[U(x) + (-U(x))]/2 = 0$, and we obtain diffusion in the absence of potential. If we put in equation (15) $\alpha \simeq \sqrt{\omega}\left[1 + \beta^2/(2\omega)\right] \to \infty$ and $\mu = \omega/\beta^2 \to \infty$, we find

$$\frac{D_{\mathit{eff}}}{D} \simeq 1 + \frac{\beta^2}{\omega}. \qquad (17)$$

The normalized effective diffusion coefficient D_{eff}/D as a function of the dimensionless mean rate of potential switching ω, for different values of the dimensionless height of potential barriers β, is shown in Figure 4. A nonmonotonic behavior for all values of β is observed. We see that $D_{\mathit{eff}}/D > 1$ for different threshold values of ω. This threshold value decreases with increasing height of the potential barrier. In the limiting case of $\beta \ll 1$, we find from

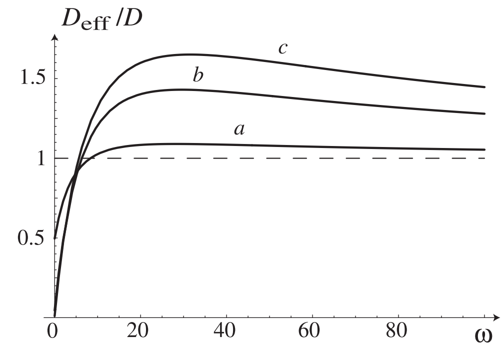

Fig. 4. The normalized effective diffusion coefficient versus the dimensionless switchings mean rate of the potential $\omega = \nu L^2/(2D)$, for different values of the dimensionless height of the potential barrier. Namely $\beta = 3, 7, 9$, for the curves a, b, and c respectively.

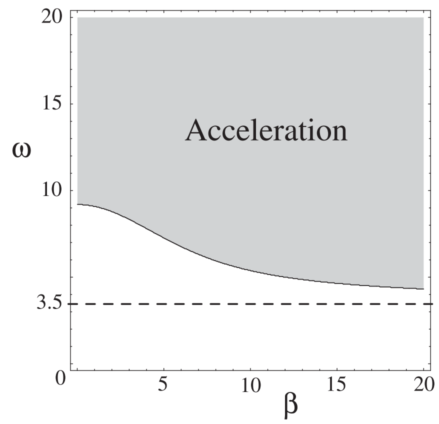

Fig. 5. The shaded area is the parameter region on the plane (β, ω) where the diffusion acceleration compared with a free diffusion case can be observed.

equation (15),

$$\frac{D_{eff}}{D} \simeq 1 + \frac{\beta^2 \cdot [(1+2\omega)\cosh 2\sqrt{\omega}}{2\omega^2 \cosh 2\sqrt{\omega}} - \frac{(4\cosh\sqrt{\omega} - 3)\,(1 + 4\sqrt{\omega}\sinh\sqrt{\omega} - 2\omega)]}{2\omega^2\cosh 2\sqrt{\omega}}. \quad (18)$$

For low barriers we obtain the enhancement of diffusion at relatively fast switchings: $\omega > 9.195$. For very high potential barriers ($\beta \to \infty$) and fixed mean rate of switchings ν, we have $\alpha \simeq \beta \to \infty$, $\mu \to 0$, and $\alpha^2\mu \to \omega$. As a result, we find from equation (15)

$$D_{eff} = \frac{\nu L^2}{7}. \quad (19)$$

We obtained an interesting result: a diffusion at super-high potential barriers (or at very deep potential wells) is due to the switchings of the potential only. According to equation (19), the effective diffusion coefficient depends on the mean rate of flippings and the spatial period of potential profile only and does not depend on D. The area of diffusion acceleration, obtained by equation (15), is shown on the plane (β, ω) in Figure 5 as the shaded area. This area lies inside the rectangle region defined by $\beta > 0$ and $\omega > 3.5$.

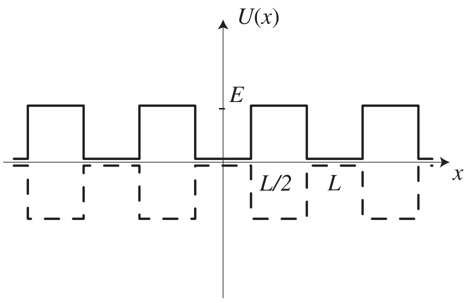

Fig. 6. Switching rectangular periodic potential.

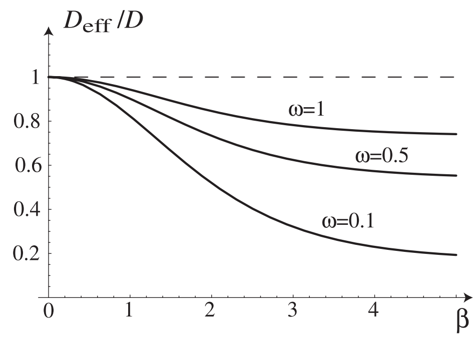

Fig. 7. The normalized effective diffusion coefficient versus the dimensionless height of potential barriers $\beta = E/D$ for different values of the dimensionless switchings mean rate $\omega = \nu L^2/(2D)$.

3.2 Switching rectangular periodic potential

For a switching rectangular periodic potential, represented in Figure 6, the main integro-differential equation (13) includes delta-functions. To solve this unusual equation we use the approximation of the delta function in the form of a rectangular function with small width ϵ and height $1/\epsilon$, and then make the limit $\epsilon \to 0$ in the final expression. As a result, from equations (13) and (14) we get a very simple formula

$$\frac{D_{eff}}{D} = 1 - \frac{\tanh^2(\beta/2)}{\cosh(2\sqrt{\omega})}. \quad (20)$$

We have slowing down of diffusion for all values of the parameters β and ω. This is because in rectangular periodic potential the Brownian particles can only move by thermal force, crossing randomly the potential barriers as in a fixed potential. The behavior of the normalized effective diffusion coefficient D_{eff}/D, as a function of the dimensionless height of the potential barrier β for different values of the dimensionless mean rate of switchings ω, is shown in Figure 7. The dependence of D_{eff}/D versus ω for different values of β is shown in Figure 8. For very rare switchings from equation (20) we obtain the same result

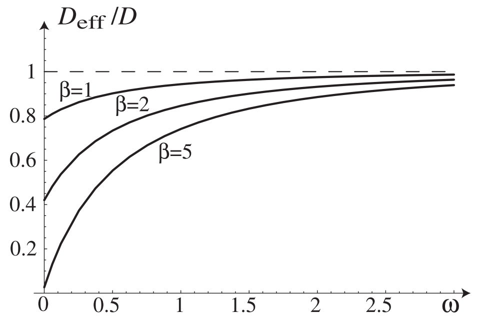

Fig. 8. The normalized effective diffusion coefficient versus the dimensionless switchings mean rate $\omega = \nu L^2/(2D)$ for different values of the dimensionless height of potential barriers $\beta = E/D$.

as for fixed rectangular periodic potential

$$\frac{D_{eff}}{D} \simeq \frac{1}{\cosh^2(\beta/2)}. \tag{21}$$

In the case of very fast flippings the effective diffusion coefficient, as for sawtooth potential (see Eq. (17)), is practically equal to the free diffusion one

$$\frac{D_{eff}}{D} \simeq 1 - 2e^{-2\sqrt{\omega}} \tanh^2(\beta/2). \tag{22}$$

For relatively low potential barriers we get from equation (20)

$$\frac{D_{eff}}{D} \simeq 1 - \frac{\beta^2}{4\cosh(2\sqrt{\omega})}. \tag{23}$$

Finally, for very high potential barriers, D_{eff} depends on the white noise intensity D and the parameter ω

$$D_{eff} \simeq \frac{2D}{1 + \coth^2\sqrt{\omega}}. \tag{24}$$

4 Conclusions

We studied the overdamped Brownian motion in fluctuating supersymmetric periodic potentials. We reduced the problem to the mean first passage time problem and derived the general equations to calculate the effective diffusion coefficient D_{eff}. We obtain the exact formula for D_{eff} in periodic potentials modulated by white Gaussian noise. For a switching sawtooth periodic potential, the exact formula obtained for D_{eff} is valid for arbitrary intensity of white Gaussian noise, arbitrary parameters of the external dichotomous noise and of potential. We derived the area on the parameter plane (β, ω) where the enhancement of diffusion can be observed. We analyzed in detail the limiting cases of very high and very low potential barriers, as well as very rare and very fast switchings. A diffusion process is obtained in the absence of thermal noise. For switching rectangular periodic potential the diffusion process slows down for all values of dimensionless parameters of the potential and the external noise.

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